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Reflection of short pulses in linear optics

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Abstract. Two methods of calculating the reflected wave generated by a short optical pulse falling on a *linear* dielectric medium are given. As examples the cases of an input sech pulse modulating a resonant carrier wave and an input delta function are calculated. At atomic densities greater than about 10^{18} atoms cm^{-3} an appreciable amount of the energy of the sech pulse is reflected. It is suggested that any nonlinear theory which ignores reflection may break down at these densities.

1. Introduction

In the last few years much interest has developed in the study of the propagation of intense, ultrashort optical pulses in a resonant medium. For an excellent survey of the theoretical work in this field we refer the reader to the review of Lamb (1971). Most of this research is based on the use of the slowly varying envelope (SVE) approximation, in which equations are derived for the envelope modulating a resonant carrier wave. Reflection and backscattering are neglected, and the envelope is assumed to be slowly varying so that higher derivatives can be dropped. Despite the many theoretical and experimental successes of this theory, some recent work (Bullough and Ahmad 1971, Bullough and Ahmad to be published) has shown that the SVE approximation is at best good only at low optical densities—namely at particle densities less than about 10^{18} cm^{-3} for dipole matrix elements of about 10^{-18} cgs units. This conclusion has been supported by some numerical studies of the exact nonlinear optics equations (Eilbeck and Bullough 1972).

In this paper we calculate the reflected wave generated by short optical pulses incident on a *linear* dielectric. This is done both for the intrinsic interest of the problem and as a test of the assumptions of the SVE approximation in *nonlinear* theory. We argue that linear theory is applicable at least to the leading and trailing edges of a nonlinear pulse, and if we find a large amount of energy reflected in linear theory it is unjustifiable to ignore reflection and backscattering in any nonlinear theory.

The advantage of linear reflection theory is that exact results can be obtained, subject to the accuracy of some simple numerical calculations. We develop here two efficient methods for the numerical calculation of the reflected wave generated by a given input pulse. The mathematical methods developed are suitable for any similar linear response theories.

The paper is set out as follows: in § 2 we review the basic results of linear reflection theory and in § 3 we describe a method suitable for treating short (picosecond) pulses which can be described as an envelope modulating a carrier wave. As an example we calculate the case of an input pulse with a hyperbolic secant envelope, a pulse of some importance in nonlinear optics (McCall and Hahn 1969, Lamb 1971). In § 4 we develop a method for evaluating the reflected wave in the case where the input pulse is extremely short ($\lesssim 10^{-15}$ s). As a simple example the case of an input δ function pulse is considered.

2. Review of linear optics reflection theory

We start with the well known equation for the reflection of a monochromatic beam of light falling on a linear dielectric medium. The dielectric is assumed to occupy the half space $z > 0$ and have a frequency dependent refractive index m. If the incident beam E_{I} has frequency ω, then the reflected beam has the same frequency; the amplitude is given by

$$E_{\mathrm{R}} = F(\omega)E_{\mathrm{I}} \tag{1}$$

where

$$F(\omega) = \frac{1-m(\omega)}{1+m(\omega)}. \tag{2}$$

For finite wave packets $E_{\mathrm{I}}(ct-z)$ and $E_{\mathrm{R}}(ct+z)$ with a spread of frequencies, we generalize (1) by integrating over all frequencies

$$E_{\mathrm{R}}(\tau_+) = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} E_{\mathrm{R}}(\omega)\,\mathrm{e}^{-\mathrm{i}\omega\tau_+}\,\mathrm{d}\omega \tag{3a}$$

$$= (2\pi)^{-1/2}\int_{-\infty}^{+\infty} E_{\mathrm{I}}(\omega)F(\omega)\,\mathrm{e}^{-\mathrm{i}\omega\tau_-}\,\mathrm{d}\omega \tag{3b}$$

where $\tau_\pm = ct \pm z$ and $E_{\mathrm{I}}(\omega)$, $E_{\mathrm{R}}(\omega)$ are now the Fourier transforms (FT) of the incoming and outgoing pulses $E_{\mathrm{I}}(\tau_-)$, $E_{\mathrm{R}}(\tau_+)$, defined by

$$E(\omega) = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} E(\tau)\,\mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\tau. \tag{4}$$

Unfortunately the integral (3) cannot be evaluated analytically, since m is in general a complicated function of ω, except in the trivial monochromatic case. Since in its original form (3) is a complicated function of (z, t) which does not make the main features and important physical consequences immediately obvious (Whitham 1965), we must in general calculate (3) numerically.

We can gain some physical insight into the form of $E_{\mathrm{R}}(\tau_+)$ by expressing the integral (3) as a convolution integral

$$E_{\mathrm{R}}(\tau_+) = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} F(\tau')E_{\mathrm{I}}(\tau_+ - \tau')\,\mathrm{d}\tau'. \tag{5}$$

From causal considerations we must have $F(\tau) = 0$ for $\tau \leqslant 0$. With a finite input pulse E_{I} it is easy to show from (5) that the reflected pulse E_{R} will have a leading edge but no trailing edge. The fact that the reflected wave can be longer than the incident pulse is an example of optical ringing: it follows that backscattering must be taking place *inside* the medium. Some calculations of the *refracted* (ie forward travelling) part of the ringing wave in *nonlinear* optics have been carried out by Burnham and Chiao (1969).

In order to evaluate (3) we need to know the function $m(\omega)$. For our purposes an adequate model is the two-state atom model (cf Bullough 1970) which gives

$$m^2(\omega) = 1 + \frac{f}{\omega_s^2 - \omega^2} \tag{6}$$

where $\hbar\omega_s$ is the energy difference between the two states; $f = 4\pi n e^2 |x_{0s}|^2 2\omega_s \hbar^{-1}$ where n is the atomic dipole density and $e|x_{0s}|$ is the dipole matrix element. In the following sections we use dimensionless units such that $c = \omega_s = 1$: with typical values for ω_s and $e|x_{0s}|$ we have for f in these units

$$f = 8\pi n \times 10^{-24}. \tag{7}$$

We give here some properties of the function $F(\omega)$ which will be needed later. From equations (2) and (6) we have

$$F(\omega) = -f^{-1}[2(\omega_s^2 - \omega^2) + f \pm 2\{(\omega_s^2 - \omega^2 + f)(\omega_s^2 - \omega^2)\}^{1/2}] \tag{8}$$

with the + sign if $\omega^2 > \omega_s^2$ and the negative sign if $\omega^2 < \omega_s^2$. From (8) we see that $F(\omega)$ has square root branch points at $\omega = \pm\omega_s, \pm\omega_c$, where $\omega_c^2 = \omega_s^2 + f$. We choose the cuts to lie on the real axis, for ω such that $\omega_s^2 \leqslant \omega^2 \leqslant \omega_c^2$. From (7) we see that the length of these two cuts $(\omega_c - \omega_s)$ will be numerically small compared to ω_s if $n \ll 10^{22}\ \mathrm{cm}^{-3}$. If the spectrum of the input pulse $E_I(\omega)$ is broad with respect to the length of these cuts then the most efficient way to calculate the integral (4) is to reduce it to an integral along the cut only: this method is described in § 4. If the spectrum of E_I is narrower than, or comparable with, the length of the cut, a direct computation of (3) as described in § 3 is sufficient.

With the choice of cuts given above, $F(\omega)$ is complex for $\omega_s^2 < \omega^2 < \omega_c^2$, and has the symmetry properties

$$F(\omega^*) = F^*(\omega) \tag{9a}$$

$$\mathrm{Im}\{F(\omega)\} = -\mathrm{Im}\{F(-\omega)\} \tag{9b}$$

$$\mathrm{Re}\{F(\omega)\} = \mathrm{Re}\{F(-\omega)\}. \tag{9c}$$

3. Reflection of a modulated sech pulse

In *nonlinear* optics a pulse of a hyperbolic secant modulating a resonant carrier wave plays an important role in the study of selfinduced transparency (McCall and Hahn 1969). In this section we consider the *linear* reflection of this pulse, with the form $(\omega_1 \ll \omega_s)$

$$E_I(\tau_-) = \mathrm{sech}(\omega_1\tau_-)\cos(\omega_s\tau_-). \tag{10}$$

However the method we describe is suitable for any shape of envelope modulating a resonant carrier wave. For our analysis we need the FT of (10) (cf Erdélyi 1954)

$$E_I(\omega) = \frac{(2\pi)^{1/2}}{\omega_1} \frac{\cosh(\frac{1}{2}\pi\omega_s/\omega_1)\cosh(\frac{1}{2}\pi\omega/\omega_1)}{\cosh(\pi\omega_s/\omega_1) + \cosh(\pi\omega/\omega_1)}. \tag{11}$$

Due to the resonant carrier wave $E_I(\omega)$ is strongly peaked about $\omega = \pm\omega_s$ and hence $E_R(\omega)$ from (3) will be peaked in this manner. $E_R(\tau_+)$ will be a slowly varying envelope modulating a resonant carrier wave and can be written in the form

$$E_R(\tau_+) = C(\tau_+)\cos(\omega_s\tau_+) + S(\tau_+)\sin(\omega_s\tau_+) \tag{12}$$

without loss of accuracy. We divide C and S into symmetric and antisymmetric parts

$$C = C_+ + C_- \tag{13a}$$

$$S = S_+ + S_- \tag{13b}$$

then equating the transforms of (12), (13) with $E_R(\omega) = E_I(\omega)F(\omega)$ we have

$$E_I(\omega)\,\mathrm{Re}\{F(\omega)\} = \tfrac{1}{2}\{C_+(\omega+\omega_s) + C_+(\omega-\omega_s) + S_-(\omega+\omega_s) - S_-(\omega-\omega_s)\} \tag{14a}$$

$$E_I(\omega)\,\mathrm{Im}\{F(\omega)\} = \tfrac{1}{2}\{C_-(\omega+\omega_s) + C_-(\omega-\omega_s) + S_+(\omega-\omega_s) - S_+(\omega+\omega_s)\}. \tag{14b}$$

Near $\omega \simeq \omega_s$, where $E_I(\omega)$ is strongly peaked, we can neglect $C(\omega+\omega_s)$ and $S(\omega+\omega_s)$ in comparison with $C(\omega-\omega_s)$ and $S(\omega-\omega_s)$. From symmetry considerations we have

$$C_\pm(\omega) = E_I(\omega_s+\omega)\,\mathrm{Re}\{F(\omega_s+\omega)\} \pm E_I(\omega_s-\omega)\,\mathrm{Re}\{F(\omega_s-\omega)\} \tag{15a}$$

$$S_\pm(\omega) = \pm E_I(\omega_s+\omega)\,\mathrm{Im}\{F(\omega_s+\omega)\} + E_I(\omega_s-\omega)\,\mathrm{Im}\{F(\omega_s-\omega)\}. \tag{15b}$$

Note that $S_+(\omega) = -S_-(\omega)$ since $\mathrm{Im}\{F(\omega_s-\omega)\} = 0$ for $\omega > 0$. From (15) $C_\pm$ and $S_\pm$ are strongly peaked about $\omega \simeq 0$ so our neglect of terms like $C_+(2\omega_s)$ is consistent. By imposing a cut-off we can invert the Fourier transforms and calculate $C_\pm(\tau_+)$ and $S_\pm(\tau_+)$ numerically. The envelope functions C and S are plotted, together with the input sech envelope E, in figure 1, for a density $n = 10^{21}$ atoms cm^{-3}. The input pulse is travelling to the right and the in-phase (C) and out-of-phase (S) parts of the reflected wave are travelling to the left. Note that the trailing edges of both components of the reflected wave are larger than that of the input wave (optical ringing). It is apparent that at this density any model of pulse propagations must take reflection and back-scattering into account.

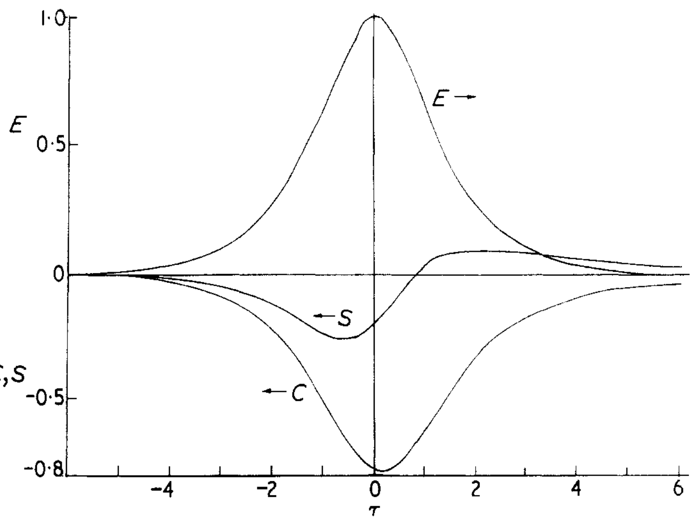

Figure 1. Plot of input sech pulse (E) and output in-phase (C) and out-of-phase (S) parts of the reflected pulse.

The fraction of input energy reflected is given by

$$\alpha = \frac{\int_{-\infty}^{+\infty} E_{\mathrm{R}}^2(\tau_+)\,\mathrm{d}\tau_+}{\int_{-\infty}^{+\infty} E_{\mathrm{I}}^2(\tau_-)\,\mathrm{d}\tau_-} \tag{16}$$

which by the Parseval formula (Titchmarsh 1948) is

$$\alpha = \frac{\int_{-\infty}^{+\infty} |E_{\mathrm{R}}(\omega)|^2\,\mathrm{d}\omega}{\int_{-\infty}^{+\infty} |E_{\mathrm{I}}(\omega)|^2\,\mathrm{d}\omega}. \tag{17}$$

We have calculated α as a function of the atomic density n from (17) by numerical quadrature. The results are displayed on a log–log scale in figure 2. Note that below $n \simeq 10^{18}$ less than 1 % of the incoming energy is reflected and hence for these densities it is reasonable to neglect the effects of backscattering and reflection. For $n \gtrsim 10^{18}$, the energy reflected is a rapidly increasing function of density, rising from 2 % reflection for $n = 10^{18}$ to 90 % reflection for $n = 10^{22}$. It is clear that in this density range any theory which ignores reflection and backscattering will be at best a poor approximation to the real world. A further interesting feature of figure 2 is that $\lg(\alpha)$ is proportional to $\lg(n)$ for $n \lesssim 10^{17}$.

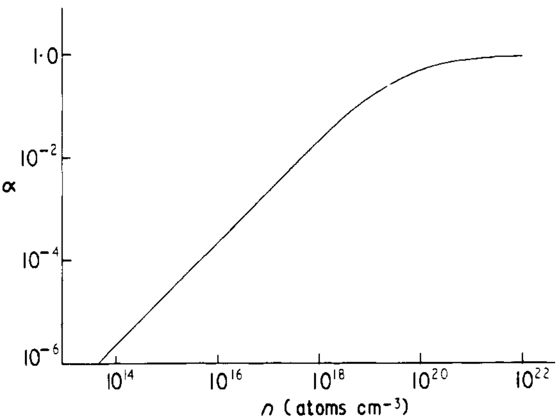

Figure 2. Fraction of energy of sech pulse reflected as a function of density (log–log scale).

Linear theory and SVE theory are different approximations to the exact nonlinear optics equations (Eilbeck and Bullough 1972). It is obvious from the results described above that these two approximations are not consistent, at least at high densities. In linear theory strong attenuation (ie $m^2(\omega) < 0$) is possible, whereas in SVE theory distortionless pulse solutions are possible (zero attenuation). To complete our arguments it is necessary to prove that the discrepancy lies in the approximations of SVE theory rather than the approximations of linear theory. A physical argument is that in a dielectric without relaxation, reflection and backscattering are the only mechanism for attenuation as opposed to dispersion: if the mechanism for attenuation is neglected the resulting theory will have no attenuation by construction. A more rigorous mathematical argument based on characteristic theory is given in the Appendix, with the density dependence of the backscattered wave clearly demonstrated. It is shown there that the equivalent condition for attenuation, namely $m^2(\omega) < 0$, is inconsistent with the assumptions of SVE theory.

4. Reflection of a δ function pulse

Generalized function theory gives a well defined FT of the Dirac δ function (Lighthill 1958). If $E_I(\tau_-) = \delta(\tau_-)$ then

$$E_I(\omega) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \delta(\tau_-)\, e^{i\omega\tau_-}\, d\tau_- = (2\pi)^{-1/2}. \tag{18}$$

The reflected wave is from (3) and (18)

$$E_R(\tau_+) = (2\pi)^{-1} \int_{-\infty}^{+\infty} F(\omega)\, e^{-i\omega\tau_+}\, d\omega \tag{19}$$

where $F(\omega)$ is defined in equation (8). The cuts of $F(\omega)$, as described in § 2, are shown in the complex ω plane in figure 3. The usual causal arguments fix the contour of integration in (19) to lie just above the real axis. This contour is shown as C_1 in figure 3. If, in (19), $\tau_+ \leqslant 0$ we can add a semicircular contour to C in the *upper* half plane which gives no contribution to the integral. Then by Cauchy's theorem

$$E_R(\tau_+) = 0 \qquad \tau_+ \leqslant 0 \tag{20}$$

which is merely a restatement of causality.

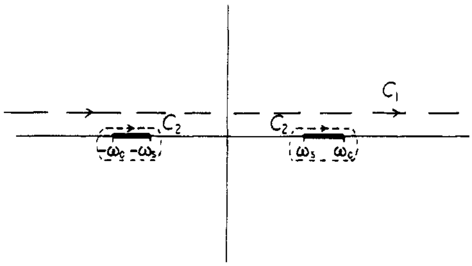

Figure 3. The complex ω plane and the cut structure of $F(\omega)$.

If $\tau_+ > 0$ we can add a contour in the *lower* half plane to form a closed contour. This contour can be deformed to get the contour C_2 shown in figure 3. From the symmetry properties of F (equation (9)) we can reduce the integral (19) round the contour C_2 to a line integral along the right hand cut

$$E_R(\tau_+) = \frac{2}{\pi} \int_{\omega_s}^{\omega_c} \mathrm{Im}\{F(\omega + i\epsilon)\} \sin(\omega\tau_+)\, d\omega \tag{21a}$$

$$= -\frac{4}{\pi f} \int_{\omega_s}^{\omega_c} \{(\omega^2 - \omega_s^2)(\omega_s^2 - \omega^2 + f)\}^{1/2} \sin(\omega\tau_+)\, d\omega \tag{21b}$$

remember that $\omega_c^2 = \omega_s^2 + f$, and that if f is small $\omega_c - \omega_s \ll \omega_s$. Some of the properties of $E_R(\tau_+)$ can be seen immediately from (21). E_R is a sinusoidal wave with a frequency spectrum having a peak at $\omega \simeq \omega_s(1 + \frac{1}{4} f/\omega_s^2)$ and a width proportional to f/ω_s. Since $\mathrm{Im}\{F(\omega)\}$ is of bounded variation, we can obtain a bound on E_R for large τ_+ by applying

Riemann's lemma (Jeffreys and Jeffreys 1962) to the integral (21). This gives

$$E_{\mathrm{R}}(\tau_+) = \mathrm{O}(\tau_+^{-1}). \tag{22}$$

Hence the *intensity* of the reflected wave goes at least like τ_+^{-2} for large τ_+.

It is only possible to integrate (22) analytically in the limit $f, \tau_+ \to 0$. (Low density, small time approximation.) For this special case we get from (21*b*)

$$E_{\mathrm{R}}(\tau_+) \simeq -\frac{\pi f}{2\omega_{\mathrm{s}}} \sin(\omega_{\mathrm{s}}\tau_+) \qquad \tau_+ > 0. \tag{23}$$

However (21*b*) is very amenable to numerical calculation. Figure 4 shows a computer calculation of the envelope of the reflected wave resulting from a unit δ function impinging on a dielectric with atomic dipole density $n = 10^{21}\ \mathrm{cm}^{-3}$.

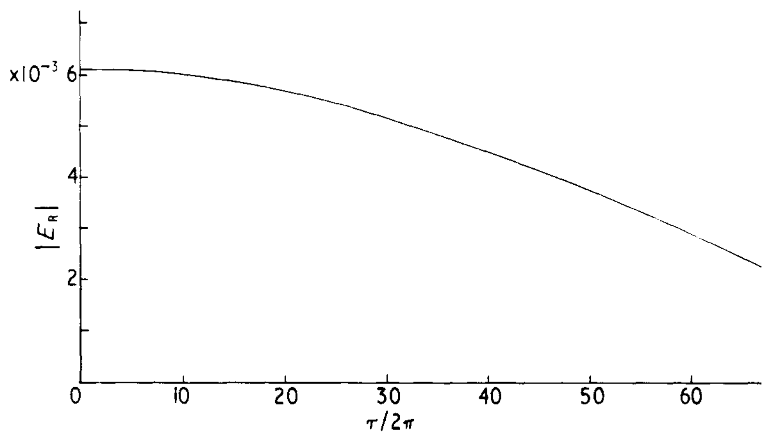

Figure 4. Envelope of reflected wave resulting from unit δ function input: $n = 10^{21}$ atoms cm^{-3}.

Generalization of this method to deal with other hypershort pulses with *simple* Fourier transforms is straightforward.

Acknowledgments

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Appendix

The basic equation common to both linear and nonlinear theories is the Maxwell wave equation, in our dimensionless units

$$E_{zz} - E_{tt} = f P_{tt} \tag{A.1}$$

where f is defined in equation (7) and $P(z, t)$ is the microscopic polarization. By introducing the magnetic field $B(z, t)$ we can write (A.1) in characteristic form (Eilbeck and Bullough 1972)

$$\frac{\partial}{\partial \tau_-}(E+B) = -\tfrac{1}{2}fP_t \tag{A.2a}$$

$$\frac{\partial}{\partial \tau_+}(E-B) = -\tfrac{1}{2}fP_t \tag{A.2b}$$

with $\tau_\pm = t \pm z$. Equation (A.2*a*) described a wave travelling in the negative z direction and (A.2*b*) a wave travelling in the positive z direction. If $E+B$ is initially zero, we can integrate (A.2*a*), under the assumption that P_t is a function of zero mean, to get

$$E+B = \mathrm{O}(f). \tag{A.3}$$

For small f (low density approximation) we can write $E+B = 0$ and the back-scattered wave is negligible. Putting $E = -B$ into (A.2*b*) gives us an equation describing a wave travelling to the right only

$$E_z + E_t = -\tfrac{1}{2}fP_t. \tag{A.4}$$

This equation is to be compared with the original wave equation (A.1) which describes waves travelling in both directions.

If we now look for solutions of (A.1) and (A.4) of the form $E = E_0 \cos(\omega t - \kappa z)$, $P = P_0 \cos(\omega t - \kappa z)$ we have from (A.1)

$$(m^2-1)E_0 = fP_0 \tag{A.5a}$$

and from (A.4)

$$(m-1)E_0 = \tfrac{1}{2}fP_0 \tag{A.5b}$$

where $m = \kappa/\omega$. The m^2 term in (A.5*a*) appears in equation (6) and gives rise to attenuation for values of ω such that $m^2(\omega) < 0$. If we use equation (A.5*b*) instead we have only a linear term in m and hence attenuation is no longer possible. Thus ignoring the back-scattered wave is equivalent to ignoring attentuation and is only justifiable at low densities.

In nonlinear theory the concept of a refractive index is no longer viable, but we argue that our general conclusion is still valid, since the distortionless pulse solutions of SVE theory are *not* exact solutions of the basic Maxwell equation (A.1) (Bullough 1971), but instead satisfy the SVE version of equation (A.4).

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